

2.1 On Minkowski spacetime (\mathbb{R}^{n+1}, η) , let (x^0, \dots, x^n) be the standard Cartesian coordinate system. Compute the induced metric on the submanifolds

$$S_{-1}^{(n,1)} = \left\{ - (x^0)^2 + \sum_{i=1}^n (x^i)^2 = -1 \right\}$$

and

$$S_{+1}^{(n,1)} = \left\{ - (x^0)^2 + \sum_{i=1}^n (x^i)^2 = +1 \right\}$$

(the latter is known as *de-Sitter* spacetime).

2.2 On Minkowski spacetime (\mathbb{R}^{n+1}, η) , let $p, q \in \mathbb{R}^{n+1}$ be two points such that $q \in I^+(p)$. Let also $\gamma_0 : [0, 1] \rightarrow \mathbb{R}^{n+1}$ be the straight line segment connecting p to q (i.e. $\gamma_0(0) = p$, $\gamma_0(1) = q$ and $\ddot{\gamma}_0 = 0$) and $\gamma : [0, 1] \rightarrow \mathbb{R}^{n+1}$ be any other *causal* curve (i.e. with $\dot{\gamma}$ either timelike or null) such that $\gamma(0) = p$ and $\gamma(1) = q$. Show that the corresponding lengths of the curves satisfy

$$\ell(\gamma_0) \geq \ell(\gamma).$$

This is a manifestation of the *twin paradox* in special relativity.

Hint: Approximate γ by a polygonal causal curve and, using the inverse triangle inequality for causal vectors, show that the line segment connecting p and q has greater or equal length to a broken line segment connecting the same points.

2.3 Let (M, g) be a smooth Lorentzian *surface* (i.e. 2-dimensional manifold).

(a) Show that for any $p \in \mathcal{M}$, there exists an open neighborhood \mathcal{U} of p and a local system of coordinates (u, v) on \mathcal{U} such that

$$g = \Omega(u, v) du dv,$$

where $\Omega \in C^\infty(\mathcal{U})$ does not vanish in \mathcal{U} (such a coordinate system is called a *characteristic* or *double null* system).

(b) Deduce that every smooth Lorentzian surface is locally conformally equivalent to an open subset of the Minkowski space (\mathbb{R}^{1+1}, η) (recall that a similar fact also holds for *Riemannian* surfaces; in that case, a coordinate system exhibiting this equivalence is called *isothermal*).

***2.4** In this exercise, we will show that there are topological obstructions to a manifold admitting a Lorentzian metric; **not** every smooth manifold admits one. To this end, let us adopt the following definition: For any Lorentzian inner product space (V, m) , we will call any 2-element set of the form $\{u, -u\}$ (where $u \in V \setminus 0$) a *line seed*. A line seed $X = \{u, -u\}$ will be called *causal* if $u \in V$ is a causal vector. We will also define the *trivial* line seed to be the pair $\{0, -0\}$. Given two causal line seeds $X = \{u, -u\}$ and $Y = \{v, -v\}$, then exactly one of the vectors $+v$ and $-v$ belongs to the same timecone as u . We will define the sum $X + Y$ as the seed $\{u + v, -u - v\}$ if u, v belong to the same time cone and as $\{u - v, -u + v\}$ otherwise. We will extend this definition to include the trivial line seed.

- (a) Verify that, with the addition operator defined above, $X_1 + X_2$ is a causal line seed if X_1, X_2 are causal line seeds or if one of them is causal and the other is the trivial line seed.
- (b) Let (\mathcal{M}, g) be a smooth Lorentzian manifold and let $p \in \mathcal{M}$. Show that there exists an open neighborhood \mathcal{U} of p and a smooth causal vector field $U \in \Gamma(\mathcal{U})$.
- (c) A smooth *line field seed* on \mathcal{M} will be an assignment of a line seed $X_p = \{U_p, -U_p\}$ in $T_p\mathcal{M}$ for each $p \in \mathcal{M}$ such that, for any $q \in \mathcal{M}$, there exists an open neighborhood \mathcal{V} of q and a smooth vector field $Y \in \Gamma(\mathcal{V})$ such that $Y(p) \in X_p$ for all $p \in \mathcal{V}$.¹ Show that \mathcal{M} as above admits a smooth *causal* line field seed.

*Hint: For this part, it might be helpful to use the fact that any smooth manifold admits a **partition of unity**: For any open covering $\{\mathcal{U}_a\}_a$ of \mathcal{M} , there exists a family $\{\chi_\beta\}_\beta$ of smooth functions $\chi_\beta : \mathcal{M} \rightarrow [0, +\infty)$ satisfying the following properties:*

- * Each χ_β is compactly supported, and its support is contained in one of the open sets \mathcal{U}_a .
- * For each χ_β , $\text{supp}(\chi_\beta)$ intersects only finitely many of the supports of χ_γ , $\gamma \neq \beta$.
- * For any $p \in \mathcal{M}$, $\sum_\beta \chi_\beta(p) = 1$.

You can then use part 2.4.b to construct a smooth causal line seed field in a neighborhood of every point in \mathcal{M} , and then use an appropriate partition of unity to “glue” these constructions together, utilising the notion of the sum of two causal line seeds from part 2.4.a.

- (d) Deduce that the tangent bundle $T\mathcal{M}$ of \mathcal{M} admits a smooth line subbundle. Can the sphere \mathbb{S}^2 admit a Lorentzian metric?

Hint: Use the fact that, for a compact manifold \mathcal{M} , if the tangent bundle admits a line subbundle then the Euler characteristic $\chi(\mathcal{M})$ of \mathcal{M} vanishes.

Bonus exercise (hard): Can you construct a Lorentzian metric on \mathbb{S}^3 ?

¹Note that, with this definition, the tangent vector U_p need not even be continuous in p ; we only require that there is (locally at least) a choice between U_p and $-U_p$ at every point p that results in a smooth vector field.